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LETTER TO THE EDITOR

Periodic self-Fourier–Fresnel functions

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Abstract. The known periodic self-Fourier–Fresnel function is comb(x, y). We show that the 2D periodic functions assembled by the Dirac comb with cell and aperture functions are the approximate self-Fourier–Fresnel functions if the cell function is compact and the cell and aperture functions are even and mutually Fourier transformable.

A self-Fourier function (SFOF) is a function that is self-reproducing under the Fourier transformation. Recently, research by Caola [1] showed that there are few functions that are SFOFs. However, Caola discovered a general recipe to construct a SFOF from any arbitrary transformable function. Along the way, Lohmann and Mendlovic [2] developed quite a number of other self-transform functions. Conversely, Cincotti, Gori and Santarsiero [3] proved that every Fourier transformable function is itself the sum of four SFOFs. Caola and Lohmann have pointed out the relevance of the results for optical applications. Moreover, Lakhtakia [4] introduced the concept of the fractal self-Fourier functions.

It is seen that these studies dealt with functions which are self-reproducing under only a single kind of transform. In this paper, we intend to study such two-dimensional (2D) periodic functions that can be self-reproduced under both the Fourier transformation and the Fresnel transformation. In coherent optics, the Fourier transform and the Fresnel transform of 2D patterns can be realized very easily, that is, a simple lens contributes the Fourier transform of a 2D pattern and the near-field diffraction of a pattern produces its optical Fresnel transform [5]. The self-Fresnel functions (SFRFs) are immediately related to self-imaging effect [6, 7], i.e. the Talbot effect, in optics, and the most useful patterns are periodic ones. It is well known that either the optical Fourier transform or the self-imaging effect has extensive applications. The 2D periodic self-Fourier–Fresnel functions (SFOFRFs) are interesting because these dual functions may provide a new basis for optical parallel processing.

The Fourier transform of a 2D function is defined by

$$\tilde{f}(u, v) = \iint_{-\infty}^{\infty} f(x, y) \mathrm{e}^{-j2\pi(xu+yv)}\, \mathrm{d}x\, \mathrm{d}y. \tag{1}$$

A SFOF is defined as any function $f(x, y)$ such that

$$\tilde{f}(u, v) = f(u, v). \tag{2}$$

The stock examples given by Caola are the Gaussian and the Dirac comb functions. However, more examples can be found in [5].

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Caola's recipe can be extended in a 2D manner to build a SFOF $F_{fo}(x, y)$ from any arbitrary transformable function $g(x, y)$:

$$F_{fo}(x, y) = g(x, y) + \tilde{g}(x, y) + g(-x, -y) + \tilde{g}(-x, -y). \tag{3}$$

All the studies show that the 2D Dirac comb function is the only periodic SFOF:

$$\mathrm{comb}(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m)\delta(y - n) \tag{4}$$

since no other periodic functions are self-reproducing under the Fourier transformation. We want to create some rules to construct the approximate 2D periodic SFOFs by virtue of the Dirac comb function. To do so, we first introduce the cell function $c(x, y)$ and the aperture function $a(x, y)$ to assemble a generalized 2D periodic function $g(x, y)$. We have two choices for organization. In the first choice the function and its Fourier transform are, respectively,

$$\begin{aligned} g(x, y) &= (a(x, y)\mathrm{comb}(x, y)) \odot c(x, y) \\ \tilde{g}(u, v) &= (\tilde{a}(u, v) \odot \mathrm{comb}(u, v))\tilde{c}(u, v). \end{aligned} \tag{5}$$

For the second choice we have

$$\begin{aligned} g(x, y) &= (c(x, y) \odot \mathrm{comb}(x, y))a(x, y) \\ \tilde{g}(u, v) &= (\tilde{c}(u, v)\mathrm{comb}(u, v)) \odot \tilde{a}(u, v). \end{aligned} \tag{6}$$

Where $\odot$ denotes a 2D convolution as

$$s(x, y) \odot t(x, y) = \iint_{-\infty}^{\infty} s(\alpha, \beta)t(x - \alpha, y - \beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{7}$$

Obviously, either of these two kinds of functions can be used to construct a SFOF in terms of the recipe indicated by (3).

As $c(x, y)$ and $a(x, y)$ are both even functions, the first two terms in (3) are enough to build a SFOF. Furthermore, it is easy to see that if $c(x, y)$ has a compact support and $a(x, y)$ has a large one, i.e. $C \ll 1$ and $A \gg 1$, where the total width of $c(x, y)$ is C and that of $a(x, y)$ is A, we have the following approximations:

$$\begin{aligned} (c(x, y) \odot \mathrm{comb}(x, y))a(x, y) &\cong (a(x, y)\mathrm{comb}(x, y)) \odot c(x, y) \\ (\tilde{c}(u, v)\mathrm{comb}(u, v)) \odot \tilde{a}(u, v) &\cong (\tilde{a}(u, v) \odot \mathrm{comb}(u, v))\tilde{c}(u, v). \end{aligned} \tag{8}$$

On this step, once $c(x, y)$ and $a(x, y)$ are a Fourier transform pair, a very interesting conclusion can be established that the functions indicated by (5) and (6) are all the approximate periodic SFOFs. In summary, they can be rewritten as

$$\begin{aligned} F_{fo}(x, y) &\cong (c(x, y) \odot \mathrm{comb}(x, y))\tilde{c}(x, y) \\ &\cong (\tilde{c}(x, y)\mathrm{comb}(x, y)) \odot c(x, y) \\ F_{fo}(x, y) &\cong (a(x, y)\mathrm{comb}(x, y)) \odot \tilde{a}(x, y) \\ &\cong (\tilde{a}(x, y) \odot \mathrm{comb}(x, y))a(x, y). \end{aligned} \tag{9}$$

The Fresnel transform of a 2D function is defined by a 2D convolution with a scaled quadratic phase function:

$$\breve{f}_z(x, y) = f(x, y) \odot \left(\frac{1}{jZ} e^{j\pi(x^2+y^2)/Z}\right) \tag{10}$$

where Z is a dimensionless distance. A SFRF is defined as any function $f(x, y)$ such that

$$\breve{f}_z(x, y) = f(x, y). \tag{11}$$

Alternatively, we have the equivalence in the frequency domain:

$$\tilde{f}(u, v) = \tilde{f}(u, v) e^{-j\pi Z(u^2+v^2)} \tag{12}$$

where the Fourier transform of a scaled quadratic phase distribution is given by

$$\int\int_{-\infty}^{\infty} e^{\pm j\pi(x^2+y^2)/Z} e^{j2\pi(xu+yv)} \, dx \, dy = \pm jZ e^{\mp j\pi Z(u^2+v^2)}. \tag{13}$$

It is apparent that $\mathrm{comb}(x, y)$ is a stock SFRF on the condition that Z is an even integer. Thus the periodic function constructed by the same cell function is again a SFRF $F_{\mathrm{fr}}(x, y)$:

$$F_{\mathrm{fr}} = C(x, y) \odot \mathrm{comb}(x, y). \tag{14}$$

Note that if $Z = 2k/K$ ($k = 1, 2, \ldots, K-1$) the self-Fresnel transform is generally a periodic function, but with different transformed cell functions. This self-imaging effect in optics is called the fractional Talbot effect [2, 6].

On the basis of these analyses, by the Fresnel transform of $Z = 2$ we can construct a SFRF from a sum of the different Fresnel transforms of a periodic self-Fresnel function:

$$F_{\mathrm{fr}} = \sum_i c(x, y) \odot \mathrm{comb}(x, y) \odot \left(\frac{1}{jZ_i} e^{j\pi Z_i(x^2+y^2)}\right). \tag{15}$$

Or, by a fractional Fresnel transform of $Z = 2/K$ we have

$$F_{\mathrm{fr}} = \sum_{k=0}^{K-1} c(x, y) \odot \mathrm{comb}(x, y) \odot \left(\frac{K}{j2k} e^{j\pi K(x^2+y^2)/2k}\right). \tag{16}$$

Now we analyse the Fresnel transformation of the generalized periodic functions marked by (5) and (6). Actually, this is a problem of the Fresnel transformation of infinite periodic functions [8]. Related to the generalized periodic function given in (6), as $A \gg 1$ we have [8]

$$[(c(x, y) \odot \mathrm{comb}(x, y))a(x, y)] \odot \left(\frac{1}{jZ} e^{j\pi(x^2+y^2)/Z}\right)$$
$$= \sum_m \sum_n \tilde{c}(m, n) a(x - mZ, y - nZ) e^{j2\pi(mx+ny)} e^{-j\pi Z(m^2+n^2)}. \tag{17}$$

Further analytical simplification seems to be impossible. The exact evaluation can be made by a numerical calculation. It is found that if $C \ll 1$ and $Z = 2$ the Fresnel transform of a periodic even function is nearly the same as itself [8, 9]. The function given in (5) is similar. On these assumptions, we can establish two approximate SFRFs in the generalized periodic function fashion:

$$F_{\mathrm{fr}} \cong (c(x, y) \odot \mathrm{comb}(x, y))a(x, y)$$
$$F_{\mathrm{fr}} \cong (a(x, y)\mathrm{comb}(x, y)) \odot c(x, y). \tag{18}$$

These two functions can also be used to construct more SFRFs in terms of the recipe given by (15) or (16).

From (2) and (12), a SFOFRF should meet the condition:

$$f(u, v) = \tilde{f}(u, v)\mathrm{e}^{-j\pi Z(u^2+v^2)}. \tag{19}$$

The known 2D periodic SFOFRF is the Dirac comb function. From all the discussions it can be recognized that the conditions for self-Fourier transformation and the conditions for self-Fresnel transformation are incompatible. In other words, except for the Dirac comb, it is unfeasible to assemble a rigorous periodic SFOFRF. So the major aim of this paper is to look for the approximate 2D periodic SFOFRFs. The two approximate periodic SFOFs have been discovered as given in (9). On the same presumptions, they are also the SFRFs (see (18)). Therefore, these two generalized periodic functions are the approximate SFOFRFs.

In conclusion, the 2D periodic functions assembled by the Dirac comb with cell and aperture functions are the approximate self-Fourier–Fresnel functions if the cell function is compact and the cell and aperture functions are even and mutually Fourier transformable. Thus a great number of approximate SFOFRFs can be constituted from the known Fourier transform pairs.

For example, we can use the Fourier transform pair of $\mathrm{rect}(x/C, Y/C)$ and $\mathrm{sinc}(Cx, Cy)$ ($C \ll 1$ and $Z = 2$) to construct an approximate 2D periodic SFOFRF as

$$F_{\mathrm{fofr}} = \left(\mathrm{rect}\left(\frac{x}{C}, \frac{y}{C}\right) \odot \mathrm{comb}(x, y)\right)\mathrm{sinc}(Cx, Cy). \tag{20}$$

For a better understanding, figure 1 shows schematically the above function in a one-dimensional distribution with $C = 0.1$.

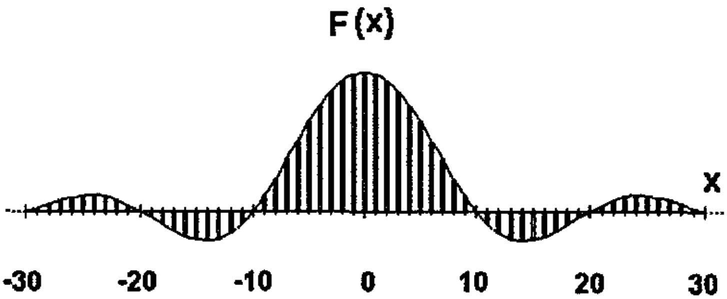

Figure 1. An example of an approximate 2D periodic self-Fourier–Fresnel function.

The optical applications of these functions have been listed in [8]. The other application is being considered for developing multiple-channel optical processing systems.

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